

$m+1$. Since $M-m > 1$,

$$(M-1)(m+1) = Mm + M - m - 1 > Mm$$

and thus the product is increased. Since the number of possibilities is finite, the stated result follows.

Solved also by Murray Barbour, B. P. Gill, H. S. Grant, P. C. Hammer, and P. A. Piza.

Editorial Note. Similar elementary reasoning shows that, k being unspecified, the maximum product which can be formed from any partition of N ($N > 1$) into positive integers is given by

$$3^{N/3}, \quad 2^2 \cdot 3^{(N-4)/3}, \quad 2 \cdot 3^{(N-2)/3},$$

according as N is congruent to 0, 1, 2, respectively, modulo 3. For any positive integer q we can write

$$(1) \quad q \leq 2^{q/2}, \quad \text{or} \quad q \leq 3 \cdot 2^{(q-3)/2},$$

according as q is even or odd. By the original result, the maximum product for any k is in the form $p^r(p+1)^s$. Without decreasing this product and without altering the sum of the factors we can replace it, according to (1), by a product of 2's and 3's. Finally, whenever the exponent of 2 exceeds 2, we can increase the product further by replacing 2^3 by 3^2 as often as possible. No further increase is now possible and the result appears as stated. The reader might care to go further and determine the maximum product of factors whose sum is N , when neither the factors nor the sum need be integers.

Generalized Simson Lines

4181 [1945, 582]. *Proposed by P. D. Thomas, Lumberton, Miss.*

Lines are drawn from a point P on the circumcircle of an equilateral triangle parallel to the three sides, thus determining six points, two on each side respectively. (1) Prove that the six points thus determined lie by threes on two straight lines. (2) If Q is the point of intersection of these two lines, find the locus of Q as P moves on the circumcircle.

Solution by R. Goormaghtigh, Bruges, Belgium. Consider a point P on the circumcircle Γ of any triangle $A_1A_2A_3$. In a system of complex coordinates having Γ as base circle (radius unity, center at origin), let t_1, t_2, t_3 , and τ be the coordinates of A_1, A_2, A_3 , and P , respectively, and let

$$s_1 = t_1 + t_2 + t_3, \quad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \quad s_3 = t_1t_2t_3.$$

If \bar{a} is the conjugate of a , then $t_i\bar{t}_i = \tau\bar{\tau} = 1$ because Γ is the base circle. If $\lambda = e^{2i\theta}$, it is easily verified that the side A_2A_3 and the line drawn from P forming with A_2A_3 the angle θ , will have the respective equations:

$$z + t_2t_3\bar{z} = t_2 + t_3, \quad z + \lambda t_2t_3\bar{z} = \tau + \lambda t_2t_3\bar{\tau}.$$

Their intersection is given by

$$z_1 = \frac{\lambda t_2 + \lambda t_3 - \lambda t_2 t_3 \bar{\tau} - \tau}{\lambda - 1}.$$

The line

$$(1) \quad \tau z + \lambda s_3 \bar{z} = \frac{\tau^2 - \lambda s_1 \tau + \lambda s_2 - \lambda^2 s_3 \bar{\tau}}{1 - \lambda}$$

is easily seen to be satisfied by z_1 , and also by the analogous points, z_2 and z_3 , on the other sides of the triangle. This line is called the Simson line, for the angle θ , of P as to the given triangle. Since the two straight lines considered in the problem are obviously the Simson lines, for the angles $\pm\pi/3$, of P as to the given equilateral triangle, the proof of the first part is complete.

The Simson line, for a second angle θ' , of P is given by (1) with λ replaced by $\lambda' = e^{2i\theta'}$. The intersection $Q(x)$ of these two lines is given by

$$(2) \quad x = \frac{\lambda\lambda'}{(1-\lambda)(1-\lambda')} \left[s_3 \bar{\tau}^2 - s_2 \bar{\tau} + s_1 - \frac{(\lambda + \lambda' - 1)\tau}{\lambda\lambda'} \right].$$

Hence the locus of Q is in general a quartic. When the triangle $A_1A_2A_3$ is equilateral, so that $s_1 = s_2 = 0$, (2) reduces to a trinodal hypotrochoid. If also, as in the present problem, $\theta = -\theta' = \pi/3$, (2) becomes the required locus

$$(3) \quad x = \frac{s_3 \bar{\tau}^2 + 2\tau}{3},$$

which is the deltoid (three-cusped hypocycloid) having its cusps at the vertices of the given triangle.

Further results of interest, related to an equilateral triangle, may be obtained from (2). The locus of intersections of Simson lines, for the angles $\pm\pi/6$, is the circumcircle Γ . For the angles $\pm\pi/4$, (2) gives the regular trifolium. If we hold P fixed while the triangle turns about its center, the locus of the intersection of the Simson lines for the angles $\pm\theta$ is a circle passing through P .

Solved also by H. E. Fettis, Ou Li, Irma Moses, and O. J. Ramler, using analysis similar to the above (and with references to Morley and Morley, *Inversive Geometry*); by J. H. Butchart, Howard Eves, and Richard Meyer, using synthetic geometry; by Claire F. Adler, W. E. Cox, D. H. Erkiletian, Jr., W. A. Rees, W. T. Short, A. Sisk, G. A. Williams, and R. H. Wilson, Jr., using standard analytic geometry; and by W. J. Robinson, using both the latter methods.

Envelope of Simson Lines

4115 [1944, 233]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

From a point P on the circumcircle of a triangle lines are drawn inclined at angles θ to the sides of the triangle and meeting them in three collinear points.