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Galileo and Leibniz: Different Approaches to Infinity

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Introduction

"God exists since mathematics is consistent, and the devil exists since we cannot prove it." In this way André Weil, one of the leading mathematicians of this century, characterized the mathematics of our time (Meschkowski 1969, p. 112; Rosenbloom 1950, p. 72).

This connection between mathematics, God and the devil is not new: I would like to recall here only the study scene in Goethe's Faust: Faust has the devil temporarily in his hand, because the pentagram is badly drawn. What is new in Weil's saying is the insight that even mathematics cannot provide complete foundations for itself. Metamathematics and Hilbert's proof theory are disciplines of our century.

When Hilbert set about showing how the antinomies of the set theory created by Cantor could be avoided without betraying mathematics, as described in his lecture "On the Infinite" in 1925, he emphasized the necessity of clarifying the nature of the infinite (Hilbert 1926, p. 188). The infinite plays a crucial role in analysis, this most ingenious product of mathematical science, being ramified in the most refined way. As Hilbert fittingly formulated, mathematical analysis is, so to speak, a pure symphony on the infinite (Hilbert 1926, p. 184).

No wonder that three years after Hilbert's lecture his pupil Hermann Weyl characterized mathematics as the science of the infinite (Weyl 1925–1927, p. 1; Laugwitz 1986, p. 233). He thus defined mathematics in a completely different way than his predecessors of earlier centuries.

In 1464, Johannes Regiomontanus, the most important German mathematician of the 15th century, defined mathematics in his introductory lecture on al-Farghānī's astronomy as the science which deals with quantities, and therefore with finite objects (Regiomontanus [1464] 1972, p. 44). This evaluation can be traced forward to the 19th century. In 1675 Jean Prestet spoke of the "mathématiques ou principes généraux de toutes les sciences qui ont les grandeurs pour objet", that is of "mathematics or general principles of all those sciences whose object are quantities" (Prestet 1675, title page). In the 18th century, the age of "universal mathematics", its representative Christian Wolff proposed the following formulation in his *Mathematical Dictionary* of 1716: "Mathematics ... is the science which measures all that which can be measured. In general it is described as "scientia quantitatum", that is, as "the science of quantities", or, in other words, of all those things which can be enlarged or reduced" (Wolff [1716] 1965, col. 863).

Considering such a definition, such a way of seeing mathematics, difficulties were inevitable as soon as the range of mathematical objects was enlarged by authors wanting to work with non-quantities. This was the case with Nicolaus Cusanus in the 15th century and Galileo in the 17th century. While the non-mathematician Nicolaus was criticized by professional mathematicians like Regiomontanus already during his lifetime, the mathematical reasoning of Galileo, the *matematico primario del Grand Duca di Toscana* (the chief mathematician of the Grand Duke of Tuscany) was handled significantly more mildly.

It is interesting therefore that, contrary to the impression often given (as I shall show) it can be established that:

1. Leibniz gave an exact definition of indivisibles, and had a clear idea of what they are.
2. He taught how to operate with them.
3. Neither Galileo nor Cavalieri thought that they are infinitely small quantities.
4. For Leibniz, indeed, an indivisible was an infinitely small quantity of indefinite size. He defined exactly what that means. Leibniz elaborated his conception of the infinite based on the notion of quantity in a conscious departure from Galileo's concept of the infinite based on the notion of non-quantities: I call these concepts the quanta concept and the non-quanta concept of the infinite respectively.

As a consequence, we have first to discuss Galileo's procedure in order to understand why and how Leibniz was able to return to the traditional concept of mathematics, to favour the quantification of the notion of indivisibles. In this respect, my main source will be his longest and most interesting mathematical treatise "On the arithmetical quadrature of the circle, the ellipse and the hyperbola" which was published only in 1993 (Leibniz 1993).

To give just one example of misleading evaluations immediately, in 1994 a monograph on the history of the number *e* appeared where we read the following affirmations (Maor 1994, p. 53, 56): "The method of indivisibles was flawed in several respects: ...no one understood exactly what these 'indivisibles' were, let alone how to operate with them. An indivisible was thought to be an infinitely small quantity." And a bit later: "The pioneers of the method of indivisibles were not clear about what exactly an 'indivisible' is." It follows from this paper that these four affirmations are historically untenable; in short, that all four are wrong.

1. Galileo's non-quanta concept of the infinite

It is precisely from the Aristotelian Simplicio that Galileo prompts Platonic praise of mathematics at the end of the first day of the *Discorsi e dimostrationi matematiche, intorno a due nuove scienze*, the *Discourses and Mathematical Demonstrations Concerning the Two New Sciences*. If he, Simplicio, could begin his studies once more, he would follow Plato's advice and would begin with mathematics. As he recognizes, it proceeds very scrupulously and does not admit anything as secure except that which it demonstrates conclusively[1] (Galilei [1638] 1965, 8:134).

Preciseness and certainty are the distinguishing qualities of mathematics – Hilbert, too, spoke in this vein (1926, p. 170) – and we shall return to this aspect in connection with Leibniz. But what was the position of these qualitites when Galileo explained his theory of the infinite and the indivisibles during this first day of discourse?

Both notions are, as Galileo admitted repeatedly (Galilei [1638] 1965, 8:73, 76, 78) incomprehensible (incomprensibili) for our finite intellect (intelletto finito); they surpass the capacity of our imagination (capacità della nostra immaginazione) (Galilei [1638] 1965, 8:83): the infinite as a result of its size (grandezza), the indivisibles as a result of their smallness (piccolezza) (Galilei [1638] 1965, 8:73). It is no surprise that he spoke of our human whims (capricci) compared to the supernatural doctrines, which are the only true and safe authorities being able to decide our controversies and which are infallible companions in our obscure and dubious paths or – rather (più tosto) – labyrinths (Galilei [1638] 1965, 8:77).

It is no accident that the appeal to the finite intellect and to theology, precisely in this connection, reminds us of Nicolaus Cusanus (Nicolaus [1440] 1967a, chapter 3) even though Galileo did not cite the Cardinal either in the *Discorsi* or in his other great works: this intellectual affinity will continue to be of great importance for us. We see though, as Galileo continued, "that human discourse cannot refrain from dealing with them"[2], i.e. the infinite and the indivisibles (Galilei [1638] 1965, 8:73). In other words, that, which we do not understand, we would least like to speak of. For according to Wittgenstein we would have to remain silent on what we cannot speak about (Wittgenstein 1978, p. 115).

Galileo developed his theory by means of four mathematical examples which implied an analysis of the linear, the two-dimensional, and the corporeal continuum as being directly connected with a physical interpretation in the form of a corpuscular theory:

1) The paradox of the Aristotelian wheel with which he began and concluded the corresponding considerations and which he treated in greatest detail.
2) The equality of certain circular rings and areas of circles which leads to the special case of the equality of the circumference of a circle with a point.
3) A comparison between the sets of the natural and the square numbers.
4) The construction of a hyperbolical point system which leads to the special case of a circle with infinite radius, which degenerates into a line.

In regard to Leibniz we are mainly interested here in the first and in the third example. The famous paradox of Aristotle's "Mechanical Problems" deals with the question why two connected concentric circles, one of which rolls along a straight line, during one revolution cover equally long straight lines in spite of their different circumferences.

Averroës had maintained that geometry cannot prove that this is the case. For this reason he was strongly criticized by Cardano (Cardano [1570] 1966, prop. 196): "Why

[1] che procedono molto scrupolosamente, nè vogliono ammetter per sicuro fuor che quello che concludentemente dimostrano.

[2] che l'umano discorso non vuol rimanersi dall' aggirarsegli attorno.

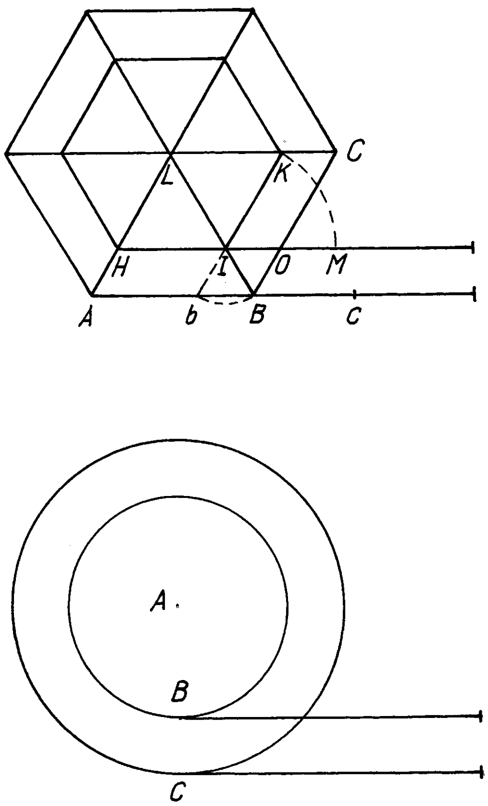

Figure 1

did he not solve the difficulty, which is exclusively mathematical and rests on evident principles?", he said in his *New Work on Proportions*. His fellow countryman Galileo, however, took the trouble to find such a proof. His fundamental idea was to understand circles as polygons with infinitely many sides (infiniti lati). This understanding enabled him to consider first the similar situation in the case of polygons with a finite number of sides, that is to say, the only case which can be understood by the human intellect. Galileo considered the problem twice. First he took the greater polygon to roll along its line and studied the resulting motion of the smaller polygon; later he took the smaller polygon to roll. Figure 1 refers to the second case. In order to analyse mathematically the finite and the infinite cases he used three pairs of notions corresponding to each other: The finite number of divisible sides[3], which are quanta, of the polygons corresponds to the infinitely many indivisible sides of the circle, these being non-quanta[4].

The most important pair of notions is "quanti/non quanti". Here, we find the key to the understanding of Galileo's theory of the infinite. Arthur von Oettingen's German translation leads the reader completely astray and enlarges the confusion by not distin-

[3] finiti lati quanti e divisibili.
[4] infiniti lati non quanti e indivisibili.

guishing between the numerative and the quantitative aspect of the notions finiti/infiniti, which can mean both finite and finitely many, infinite and infinitely many. As to the "non quanti" it is not a matter of "infinitely small". Only Goldbeck (1902) did not commit this error. But even he merely called the translation of "non quanti" by "infinitely small" questionable. All other modern authors like Bolzano (1851, p. 89–92), Lasswitz (1890, 2:45), Crew and de Silvio (1914), Mieli (1938, p. 218), Drabkin (1950, p. 185), Braga (1950, p. 306), Clavelin (1959), Struik (1969, pp. 201–207), Baron (1969, p. 117) have made the same mistake. Braga, for example, said explicitly: "Indivisible means the 'actual infinitely small'[5]". In truth, it is a matter of non-quanta. Indivisibles are non-quanta.

Galileo's argumentation as well as his terminology refer loudly and clearly back to Nicolaus of Cues. In spite of this, the terminological connection has not been investigated before (Goldbeck 1902). In his work "On the Instructed Ignorance" (chapter 14) Nicolaus recommended the ascent from the quantum triangle ("triangulum quantum") to the non-quantum triangle ("triangulum non quantum"), in order to better understand the statement that the infinite line is a triangle: If the base angles of a quantum triangle are assumed to be zero degrees, the triangle degenerates into a line: "As a consequence, you will be able to help yourself by making this assumption which is impossible in quanta (he meant that the angle opposite to the base becomes 180°) by ascending to non-quanta. There it is, as you see, entirely necessary, which is impossible with quanta" (Nicolaus [1440] 1967a, p. 17). Nicolaus underlined explicitly in "De coniecturis" (On Conjectures), that the coincidence of the contraries – of quantum and non-quantum, of straight and curved, of line and triangle – cannot be reached in mathematics. For that reason its proofs are the most reasonable and the most true according to reason (Nicolaus [ca. 1444] 1967b, p. 147).

Galileo, in proceeding in such a way, posed three decisive questions to himself:

1. How can we manage the transition from a finite number to infinitely many sides, from quanta to non-quanta, from divisibles to indivisibles?
2. What happens during this transition?
3. What are the differences between the respective notions of the three pairs mentioned in the first question?

1. The realization of the transition. The transition cannot be realized step by step by continued divisions of the divisible (Galilei [1638] 1965, 8:82), because such successive divisions do not lead to a last division. But the last division (ultima divisione) is exactly the division into the infinitely many indivisibles looked for, that is to say, into the infinitely many non-quanta. As a consequence, Galileo made use of an artifice (artifizio), which his interlocuters were requested to concede him (Galilei [1638] 1965, 8:93): The whole infinity must be distinguished and resolved at a single stroke[6]. Galileo's procedure took into account the fact that a limit is not a special element of the sequence which converges

[5] 'indivisibile' vuol dire 'infinitesimo attuale'.
[6] di distinguere e risolvere tutta la infinità in un tratto solo.

towards this limit, and what is more, that a transfinite number is not a special case of the finite real numbers (Toth 1987, p. 176).

2. The occurrences during the transition. The transition leads into a new area, which was closed for mathematics until then, where the prevailing relations and rules lose their validity, and what is worse, their applicability. This difference should not be obscured. The transition is accompanied with metamorphoses ("metamorfosi") (Galilei [1638] 1965, 8:85). When a terminated quantity ("quantità terminata") proceeds ("trapassar") to the infinite, "it meets with an infinite difference, what is more, with a vast alteration and change of character" (Galilei [1638] 1965, 8:83). And indeed: The corresponding notions result from each other by logical negations: finite becomes infinite, divisible becomes indivisible, quanta become non-quanta.

It is noteworthy that here, as when he compared the number sets, Galileo did not contrast finite with infinite, but terminated with infinite. This distinction between terminated and finite is to be found already in Nicolaus (Nicolaus [1440] 1967a, chap. 6) and we shall come across it again in Leibniz. It has been lost, however, by the translation of the two notions "terminated" and "finite" ("finita, terminata") by one and the same word German word "endlich" ("finite").

3. The differences between the respective notions of the three pairs. Quanta are divisible, they can be applied as measuring units and they are comparable in arithmetical operations. Non-quanta are not divisible, cannot be applied in such a way, and are not comparable. Neither the German nor the English translation does justice to Galileo's careful formulations:

a) Relating to measurement. Let us consider the case that the larger polygon rolls along its line.

According to Galileo in the case of polygons of 100,000 sides, the straight line ABC etc. (see Fig. 1) which is traced out is traversed and measured by the perimeter of the larger polygon[7]. This straight line is equal ("è eguale") to that straight line, which is measured by the 100,000 sides of the smaller polygon and in the course of which just as many empty spaces as sides are inserted. In the case of the circles, Galileo only said that the infinitely many sides have traversed the straight line[8]. The notion "measured" ("misurata") could no longer be mentioned. The straight line traversed by the smaller polygon is equalized with that of the greater circle ("esser pareggiata") by inserting infinitely many vacua non-quanta ("vacui non quanti") between the infinitely many sides. "By imagining that the straight line is resolved into its parts which are non-quanta, that is into their infinitely many indivisibles, we can conceive them as drawn apart "in immenso". The German version (Galilei [1938] 1973, p. 25) translates this as "ins Weite", "widely", the English as "indefinitely" (Struik 1969, p. 201). Braga even paraphrased "all' infinito", "to infinity", as if immenso and infinito were equivalent (1950, p. 314). A bit later Galileo said that we could conceive of the first infinitely many components, since these are non-quanta, as being drawn apart "in spazio immenso" by inserting infinitely many vacua, which are

7 linea passata e misurata.
8 linea passata da gl'infiniti lati.

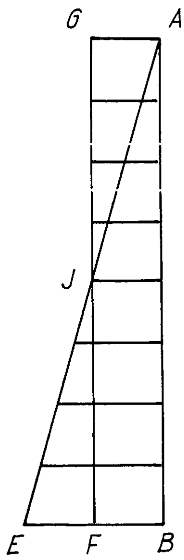

Figure 2

likewise non-quanta. The German text translates this as “zusammensetzen zu sehr großen Körpern”, “to compose into very large bodies” (Galilei [1638] 1973, p. 25), the English again as “indefinitely extended” (Struik 1969, p. 202). The intended sense, however, may well be that the indivisibles are separated from each other by a non-measurable space and that this is because vacua non-quanta, “vacui non quanti”, are inserted between them: A non-quantum occupies a “non-measurable space”.

b) Relating to calculation. Non-quanta cannot be added on account of their non being quantities. When deducing the law of falling bodies Galileo said that the aggregation (“aggregatus”) of all parallels contained in the quadrilateral A G F B are equal to the aggregation of all parallels contained in the triangle A E B (Galilei [1938] 1965, 8:208). For those in the triangle I E F are equal to those contained in the triangle G I A, while those included in the trapezium A I F B are common.

It is a gross misconception to translate aggregation by the German “Summe” (Galilei [1938] 1973, p. 158; Szabó 1987, p. 50) or by the English “sum” (Struik 1969, p. 209), which was usual even for Cavalieri’s “congeries” until very recently (see, however, Beeley 1995, p. 125; Koyré 1973, p. 341f.). In 1676 Leibniz defined the terminus according to Galileo’s view: “quod reapse divisum est seu aggregatum”, “what is really divided or what is an aggregation” (Leibniz 1980, p. 503).

c) Relating to comparisons. Non quanta cannot be compared because, unlike quantities, they lack the property of being measurable. As a consequence, it is not convenient (inconveniente), to ascribe characteristics to the infinites, to the “infiniti”, which we ascribe to “finite” and “terminated” things, “cose finite e terminate”. When he compared the sets of natural and of square numbers with each other, Galileo correspondingly observed that the attributes equal, greater, smaller do not occur between infinites, but only between “quantità terminate”, between terminated quantities (Galilei [1938] 1965, 8:78). Once more he contrasted infinite with terminated; he spoke only of infinites, not of infinite quantities, and on the other hand only of terminated quantities.

An "infinite quantity" ("quantità infinita") would according to Galileo's conception actually be a "contradiction in terms", because an infinite lacks precisely those properties which characterize a quantity. Galileo's application of a one-to-one correspondence between two number sets led him neither to a characterization of the infinite sets through comparison with the finite according to Dedekind's definition, nor to a classification of infinite sets amongst themselves. As in Nicolaus of Cues there are no different infinites, and the Cardinal certainly provided the model for Galileo's identification of the infinite number – on condition that there is such a number – with the unity (Galilei [1638] 1965, 8:83; Goldbeck 1902, pp. 77, 99).

Correspondingly, the Euclidean axiom "The whole is greater than a part" is not invalidated in the sense that the logical opposite is valid in the domain of infinite sets, that is, that an infinite set is smaller than or equal to one of its parts. Rather it is invalidated in the sense that it cannot be applied there, simply because there are no quantities which could be compared.

2. Leibniz's quanta concept of the infinite

Already in 1672 Leibniz had studied Galileo's "Discorsi" critically. This is proved by the "Accessio ad arithmeticam infinitorum" (The introduction to the Arithmetic of Infinites) as well as by his corresponding reading-notes which were written down in his first years in Paris (Leibniz 1980, number 119).

These writings make clear that, from the Parisian period onwards Leibniz's opinion differed fundamentally from Galileo's, and consequently also from Cusanus's regarding the infinite. The most comprehensive discussion of the theory of indivisibles and the most diligent foundation of his own theory of the infinitely small and large is represented by his treatise on infinitesimal geometry "On the Arithmetical Quadrature of the Circle, the Ellipse and the Hyperbola" which he wrote at the end of the sojourn in Paris and which was published in 1993 for the first time (Leibniz 1993). I shall refer mainly to this work.

The universal validity of the Euclidean axiom "The whole is greater than its part" was for Leibniz beyond question right from the start. Therefore, in the "Accessio" he considered the identification of an infinite number – on condition that there is such a number – with zero, and not, as Galileo had done, with unity. For only if we assume the existence of an actually given infinite number are we led to the Galilean paradox of the number sets. This is why such a number has to be identified with nothing, i.e. with zero. At the same time this proves Leibniz's lifelong rejection of the actual infinite in mathematics and therefore of the theory of indivisibles outlined by Galileo and Cavalieri (Breger 1990, p. 59).

According to his law of continuity the same rules must hold in the finite as in the infinite, as he wrote to Varignon in 1702 (Leibniz [1849–1863] 1962, 4:93). But to what mathematical infinite does this refer? Leibniz answered this question in his treatise on the quadrature of conic sections.

The sixth theorem of this treatise serves to lay the foundations of the method of indivisibles in the soundest way possible (firmissime), and it supplies the proof of the method of indivisibles which enables us, as Leibniz said (Leibniz 1993, p. 29) to find the areas of spaces ("areas spatiorum") by means of "sums of lines". He explained these sums as sums of rectangles with equal breadths of indefinite smallness ("indefinitae parvitatis") (Leibniz 1993, p. 39). Those who do not take these precautions, are easily misled by the method of indivisibles. In other words, Leibniz defined "lines", that is linear indivisibles as infinitely small rectangles, that is as variable quantities: his starting point was a quantification of the notion of indivisibles.

The theorem is, he said, "spinosissima", which can mean the most thorny or the most subtle and sophistical. And here it is demonstrated "morose", that is "pedantically", "overly carefully" (not "with conventional rigour", as Lucie Scholtz (1934, p. 16) translated with unintentional irony, because in that case conventional rigour would be identified with pedantry), that the procedure of constructing certain rectilinear stepped spaces and in the same way polygons can be continued to such a degree that they differ from each other or from curves by a quantity ("quantitate") which is smaller than any given quantity. This, he added, is usually assumed – Archimedes is meant – in most cases. The decisive notion is quantity (Leibniz 1993, p. 24).

This is achieved even when the number of inscribed steps or sides of polygons remains finite ("numero inscriptionum manente finito tantum") (Leibniz 1993, p. 28). He who does not want "supreme rigour" ("summus rigor"), would do better to omit the theorem at the beginning and read it only when the whole subject has been understood in order that its "excessive exactness" ("scrupulositas") does not discourage the mind from other far more agreeable things by making it become weary prematurely.

Here again is the exactness of mathematics, which Galileo had already praised. Leibniz showed how calculating with the infinite and how the method of indivisibles can be provided with supreme rigour and certainty through finite means. This deserves to be emphasized because the contrary is maintained without thinking in a monograph concerning the history of the number *e* which appeared recently (Maor 1994, p. 53). But Leibniz also saw the disadvantages of such a procedure. "I would have preferred to omit this theorem", he added in a comment on it, "because nothing is more alien to my mind than the scrupulous attention to minor details (scrupulosae minutiae) of some authors which imply more ostentation than utility. For they consume time, so to speak, on certain ceremonies, include more trouble than ingenuity and envelop in blind night the origin of inventions which is, as it seems to me, mostly more prominent than the inventions themselves" (Leibniz 1993, p. 33).

The demonstrations of the forty-five theorems that follow are based on this concept of infinitely small and infinite quantities, that is, quantities which are, according to his definition, positive, but smaller than any given quantity or larger than any given quantity. The decisive aspect is that they are quantities, fictive ones certainly, because they were introduced by a fiction, but quantities nevertheless. It does not matter whether they appear in nature or not, because they allow abbreviations for speaking, for thinking, for discovering, and for proving. In Leibniz Galileo's non-quantities have become quantities, which therefore mathematics can handle and this means, above all, with which it can calculate; in Leibniz, too, the certainty of mathematics is emphasized (Leibniz 1993,

p. 69). Leibniz thereby establishes an arithmetic of infinites which differs throughout from the work of John Wallis under the same title.

Later on Leibniz pointed out his standpoint again and again: "We need not take here the infinite in the strict sense of the word", he wrote in 1701 in the *Mémoires de Trévoux*, referring to the Marquis de l'Hôpital, in order to emphasize the certainty of the method of using infinitely large and infinitely small quantities (Leibniz [1849–1863] 1962, 5:350). In 1706 Bartholomäus des Bosses (Thiel 1982, p. 48f.) raised some doubts with regard to such a use of the infinite. Leibniz repeated that the way of talking of the infinite as quantity is non-real and merely supported by an analogy.

We are reminded throughout of Galileo's description of a "similar discourse" ("simil discorso") (Galilei [1638] 1965, 8:95) and of his remark that "the definitions of mathematicians are an imposition of names or do we want to say abbreviations of speech"[9] (Galilei [1638] 1965, 8:74). But while Galileo had said that the non-quanta are incomprehensible ("incomprensibili"), Leibniz underlined that proofs of theorems which are based on these fictive quantities, make it possible to have a clearer comprehension of these things ("clariores de his rebus comprehensiones") (Leibniz 1993, p. 35). Leibniz applied the same word as Galileo in respect of etymology. Quantities cannot only be handled mathematically, they can also be understood. For firstly, such proofs do not need both inscriptions and circumscriptions, but use only one of these two possibilities. Secondly, they simply show that two quantities are equal whose difference is infinitely small.

Leibniz did not conceal how slippery calculation with the infinite is ("quam lubrica sit ratiocinatio circa infinita") (Leibniz 1993, p. 67), if it is not guided by the thread of a proof. His own example is a theorem concerning the hyperbola: In a simple analytical relationship

$$y^n \cdot x^m = a$$

the ratio of the zone between two ordinates, the arc of the curve and the axis and the so-called "conjugated zone" between the two corresponding abscissas, the same arc of the curve and the conjugated axis, is $n{:}m$. In the case of the hyperbola, the equally hatched zones are equal, because $n = m = 1$:

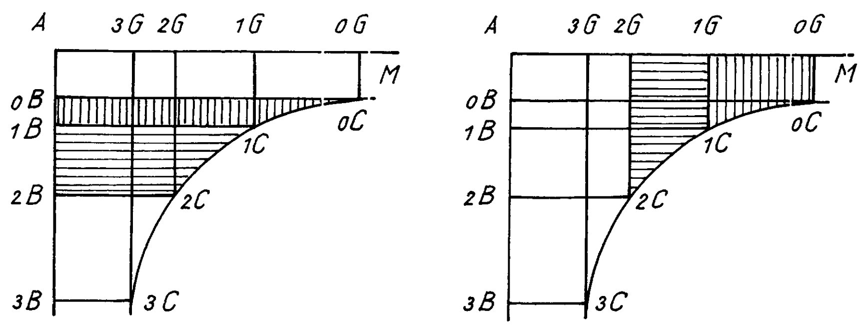

Figure 3

[9] sono una imposizion di nomi, o vogliam dire abbreviazioni di parlare.

This applies to any two arbitrary conjugated zones. Hence, all horizontal zones up to A fill the area 2C2BAM2C, all vertical corresponding zones fill only the area 2C2GM2C (Knobloch 1994). Hence a contradiction with the Euclidean axiom on the part and on the whole seems to emerge, because a part equals the whole. This result, however, was absurd in Leibniz's opinion; it came about only because the consideration proceeds from a last abscissa which does not exist in reality, or in other words, because the indivisible, the point, is identified with an infinitely small quantity, even though both notions differ basically from each other. This example shows that one cannot always jump from a certain property of perpetually cut off finite pieces to the property of the whole infinite space. Leibniz said: "We cannot jump, leap over" ("non posse prosiliri") (Leibniz 1993, pp. 63, 67). In order to reach the asymptote, in order to move from the finite pieces of area to the infinite area, a leap is called for: this corresponds to Galileo's explanation that the infinite cannot be reached by successive progression, but only at a stroke ("in un solo tratto"). And when Leibniz said that great caution and distinction ("distinctio") are needed in leaping away to infinite areas (Leibniz 1993, p. 63), there is a linguistic parallel to Galileo's "to distinguish the infinity" ("distinguere l'infinità").

Yet a fundamental conceptual difference separates the two authors: indivisibles in the strict sense of the word, points, were no more objects of mathematics for Leibniz than the unbounded, infinite asymptote. For neither indivisibles in this sense nor points are quantities, and such a line could not even be thought of as quantity in a fictive way. As a consequence, he distinguished between two infinites, the bounded infinite straight line, the recta infinita terminata, and the unbounded infinite straight line, the recta infinita interminata. He investigated this distinction in several studies from the year 1676. Only the first kind of straight lines can be used in mathematics, as he underlined in his proof of theorem 11. He assumed a fictive boundary point on a straight half-line which is infinitely distant from the beginning: a bounded infinite straight line is a fictitious quantity. Leibniz created the logical opposite of the modern conception of the finite, but unbounded universe by establishing the bounded or terminated infinite. In this respect, the difference between terminata and finita, between the terminated and finite, plays a crucial role in the analysis of the infinite, in spite of all the contrasts with Galileo.

Leibniz had put the handling of infinitesimals on a secure finite basis by means of his quanta concept of this notion. In 1821, Cauchy aimed with good reason at rigour and simplicity by means of infinitely small quantities (Cauchy [1821] 1990) and dealt with the notions of "grandeur" and quantity in a voluminous appendix. It was not their fault that in 1842 Kummer defended the following thesis in his inaugural address in Breslau:

> Differentials are quanta and they are not quanta.
> Differentialia sunt quanta, et non sunt.
> (Meschkowski 1967, p. 59)

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