THE BEGINNINGS OF ANALYTIC GEOMETRY
IN THREE DIMENSIONS

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1. Statement of the problem. In Section 528 of Jowett's translation of Plato's Republic we find the writer complaining of "the ludicrous state of solid geometry." His objection is to the fact that in his time, as in our own, geometry was far more developed in the plane than in a space of three or more dimensions. The reasons for this are fundamental. The fact that the data are more simple in the plane than in three or more dimensions, or that fewer independent variables are involved, not only makes the subject easier at the outset, but curiously enough, when we come to the higher reaches, we can introduce powerful techniques in two dimensions which are not available in spaces of greater dimensionality. Naturally Plato knew nothing of all this. His own efforts to improve the situation by studying regular and semi-regular solids produced no great result. What is really curious is that there was a similar lag in the development of analytic geometry. We shall see that a surprisingly long time passed from the publication of Descartes Géométrie until something really effective was accomplished for the algebraic geometry of space.

But first we must come to agreement about definitions, for there are different views as to the meaning of the principal terms. It is my own view that what we mean by analytic geometry is essentially the study of loci by means of equations connecting determining coördinates, regardless of the notation employed. Consequently, I hold that, strictly speaking, plane analytic geometry was introduced by the Greeks in their study of the conic sections. In Apollonius, for example, the study of the ellipse is based on the fact that the square of an ordinate is a constant multiple of the distance from its foot to the ends of the diameter in the conjugate direction. Logically then, I should maintain that the beginnings of analytic geometry in three dimensions are to be found in Archimedes' study of conoids and spheroids, where the ellipsoid of revolution, for instance, is the surface whose equation we should write

\[ y^2 + z^2 = \frac{b^2}{a^2} x(2a - x). \]

However, if we look into the matter further, we find that Archimedes did not write this equation or anything like it. He used the properties of the circles which were in planes perpendicular to the axis, and of the ellipses in planes through that line.

The delay in extending algebraic methods to three dimensions appears the more surprising when we reflect that, after all, we live in a three-dimensional physical world, and the advantage of having a frame of reference to which we can refer physical objects, saute aux yeux. But probably this advantage was not fully understood until Newtonian mechanics made familiar the ideas of instantaneous velocity and acceleration.
2. Descartes and Fermat. In looking for the beginnings of three-dimensional analytic geometry it is natural to turn to the two great Frenchmen who introduced it in the plane. In fact, it is sometimes asserted that Descartes was the first to envisage three-dimensional analytic geometry. This seems to me to be scarcely the case. At the end of the second book of his *Géométrie*, which first appeared in 1637, we find him writing:

Au reste je n'ai parlé en tout ceci que des lignes courbes qu'on peut décrire sur une superficie plate, mais il est aisé de rapporter ce que j'en ai dit à toutes celles qu'on sauroit imaginer être formées par le mouvement régulier des points de quelque corps dans un espace à trois dimensions.

It is to be noticed first that he is interested in curves, not in surfaces. His method is to project his space curve orthogonally on two mutually perpendicular planes. The space curve is thus the total or partial intersection of two cylinders whose elements are mutually perpendicular. But at this point he makes a curious slip:

Même si on veut tirer une ligne droite qui coupe cette courbe au point donné à angles droits, il faut seulement tirer deux autres lignes droites dans les deux plans, une en chacun, qui coupent à angles droits les deux courbes qui y sont aux deux points où tombent les perpendiculaires de ce point donné, car ayant élevé de deux autres plans, un sur chacune de ces lignes droites qui coupe à angles droits le plan où elle est, on aura l'intersection de ces deux plans pour la ligne droite cherchée.

This says that if we wish a normal to a space curve, we have but to construct normals to the two corresponding plane curves at the corresponding points, and pass through each of these a plane perpendicular to the plane of the curve; but it is easy to show by a particular example that we do not in this way get a normal to the space curve. I can not help wondering whether Descartes did not mean that we may get the tangent to the space curve by erecting planes tangent to the two plane curves, and orthogonal to their planes.

When we speak of Descartes in connection with analytic geometry we naturally think of the man who, excluding the Greeks, was the co-discoverer, Pierre Fermat. Here we meet a curious situation. In 1643, he sent to de Cercavi a manuscript entitled, *Ad locos ad superficiem Isogoge*. Apparently this was first published by Tannery and Henry in the first volume of their edition of Fermat's works which appeared in 1891. In a note at the bottom of page 111 the editors write:

Fermat, dont le point de départ est le livre d'Archimède *De conoidibus et spheroidibus* a bien reconnu la nécessité de généraliser la notion de la surface cylindrique, aussi que celle des conoïdes (paraboloïdes elliptiques et hyperboloïdes à deux nappes) et sphéroïdes (ellipsoïdes) d'Archimède, qui n'avait traité que les surfaces de révolution. Mais il n'a pas soupçonné l'existence du paraboloïde hyperbolique, ni l'hyperbolotoïde à une nappe.

This clearly suggests that he was on the look out for quartic surfaces which were not surfaces of revolution. In his third lemma in this communication he writes:

Si superficies quaepiam planis quotlibet in infinitum secetur, et communis sectio omnium secantium planorum et dictae superficii sit quandoque circulus quandoque ellipsis et nihil praeteria, superfies illa erit spheroidis.
This says that if all sections of a surface are either circles or ellipses the surface is a spheroid. Now if this means that every such surface is a spheroid in the Archimedean sense it is clearly wrong, as the general ellipsoid fulfils the condition. If it is a definition of an ellipsoid it is correct. But why does he call it a "lemma"? An exactly similar difficulty occurs in the case of the conoid. A possible explanation is that the word "superficies" means "surface of revolution," but then the note of Tannery and Henry is wrong. I confess that the whole situation is puzzling to me.

The principal use which Fermat makes of these lemmas is to prove propositions such as the following: Given a number of planes and a point $p$ which moves in such a way that the sum of the squares of its distances to these planes, each measured in a direction which makes a given angle with the normal to the plane, is constant; the locus of $p$ is a spheroid.

Let us see how this locus will meet a general plane $\pi$. The distance from $p$ along a line which makes a fixed angle with the normal is the normal distance divided by the cosine of the angle. The normal distance is the distance to the intersection of the plane with $\pi$, multiplied by the sine of the angle of the two planes. Hence the intersection with $\pi$ is a curve such that the sum of constant multiples of the squares of its distances from given lines is constant, and such a curve was known to be an ellipse. Hence the surface is a spheroid.

I might say, in conclusion, that previously another mathematician might have given a definition of spheroid which justifies Fermat's lemma, but I know of no such before 1643.

3. Wallis. John Wallis in his Tractatus de sectionibus conicis, Oxford, 1655, reaches the quadric surfaces in the following very different fashion. He had been deeply influenced by Cavalieri's method of indivisibles. He begins by showing that triangles of equal bases and altitudes are equivalent, that is, they have the same area, because they may be conceived as being composed of the same infinite set of line segments parallel to the bases, but differently placed. The area is not changed even if the lateral sides of the triangle are curved. We might start with any triangle, move one vertex any distance we please parallel to the base, connect the new vertex with one end of the base by any simple continuous curve, then move the infinite number of segments parallel to the base, parallel to themselves to new positions where one end is on this new curve. As the triangle is conceived as the sum of all of these segments, its area has not changed. The segments are conceived as infinitely thin rectangles.

Wallis extends this same process to three dimensions. Let us take as base a plane convex figure which is symmetrical with regard to its center $C$. Let a line through $C$ meet the base curve in $B$ and $S$, and let $V$ be a point outside the plane of the base. We obtain a cone by connecting $V$ with every point of the base curve. But Wallis describes the situation differently. Through each point of the axis $VC$ we pass a plane parallel to the base plane, and in it describe a figure similar and similarly placed to the base. Let $y$ be the distance in such a plane between the intersections with $VC$ and $VB$. The ratio of the area in this
plane to the base area is $y^2/CB^2$. Wallis next replaces the lines $VB$, $VS$ by a conic whose tangent at $V$ is parallel to the base. We cut this by a set of planes parallel to the base plane, the ordinate $y$ having the same meaning as before. In each such plane we have a closed figure, similar to the base figure, symmetrical with regard to the intersection with $VC$, a diameter of the conic; the ratio of the area in this plane to the base area is given by the previous equation. This figure he calls a "pyramidoid" or "conoid," parabolic, elliptic, or hyperbolic, as the case may be. At the end of the article, pages 111 and 112, the words, ellipsoid, paraboloid, and hyperboloid are introduced. In a plane through the axis of the original conic we have a figure similar to this conic, all ordinates being altered in the same ratio. He also says that in a general plane cutting the axis, the section will be a conic, but this is only true when the base is a conic, which he does not explicitly assume. Kötter interpreted Wallis, proceeding much more simply, by saying the surface is generated by a series of similar and similarly placed central conics, in parallel planes, whose axes are the double ordinates of a given conic [1]. This is perfectly true when the base curve is a conic, and we can obtain the general quadric surface in this way, but it is less general than what Wallis actually says, and Kötter apologizes because in the figure for a parabolic pyramidoid the base figure has five sides when he wanted it to have six.

Wallis is particularly interested not in the surface he generates in this way, but in the volumes. The integrations he performs ingeniously. Suppose the original conic is a parabola, leading to

$$y^2 = kx; \quad \int y^2 dx = k \int x dx.$$

To find this integral, suppose we have an isosceles right triangle on which stands a right prism. The area of the section in a vertical plane at a distance $x$ from the opposite edge can be written so that this integral gives us the volume of the prism, which, however, we know otherwise.

When the original conic is an ellipse, we know from Apollonius that

$$y^2 = \frac{b^2}{a^2} (2a - x).$$

Suppose then, that we have a tetrahedron whose vertices are $(0, 0, 0)$ $(0, 0, c)(2a, d, 0)(2a, -d, 0)$. The area of a rectangular section determined by the plane, $x = a$ constant, is $cdx(2a - x)/2a^2$, so we get the volume of a pyramidoid or conoid from that of a pyramid. It is worth mentioning that the scheme of finding an integral by comparing the volume of a solid calculated by two different subdivisions is found in Pascal. Fermat, writing in 1643, certainly got no help from Wallis who wrote in 1655.

I must mention one other writer who is supposed to be the first to have introduced Cartesian geometry in three dimensions. I refer to Johann Bernoulli. Here is the opinion of Beniamino Segre:
Poco appreso Joh. Bernoulli, lettere al Leibniz di 4 dicembre 1697, del 26 agosto 1698 a del 6 Febrario 1725, utilizza le coördinate per la determinazione delle geodetiche su certi tipi di superficie [2].

This statement is not absolutely correct. In the first two places mentioned there is no discussion of coördinates. The third is more helpful:

Intello per superificem datam, cujus singula puncta determinatur (sic ut lineae curva puncta) per ordinares tres x, y, z quarum relatio data aequatione exprimeretur; sunt autem tres illae coördinatae nihil aliud, quam tres tres rectae ex qualibet superficie curvae puncto perpendiculariter ductae in tres planæ positione data, et se mutuo ad angulos rectos secantis [3].

Leibniz' reply appears on the next page:

Doctrinam de aequationibus localibus trium coordinatarum, seu de locis vere solidis, olim aggregati coepi, eorumque intersectiones sive curvas etiam non planas, sed probei non vacavit.

This is perfectly clear, but I must point out that the date is 1715, by which time, as we shall see, the method had been publicly explained. In 1728, Johann Bernoulli wrote to Klingsternia:

Sit $AE = x, EB = y$ et a puncto $B$ erigi intelligentis recta $Bb = z$ normalis and planam $AEB$ et superficie curvae occurrens in $b$ et detur aequatio quaevis exprimans relationem trium coordinatarum $x, y, z$ qua relatione natura superficie determinatur [4].

I should also mention in this connection a word by Clairaut:

Je ne crois pas cette matière moins neuve que celle des courbes à double courbure, et je ne sais de connu sur ce sujet, que la façon d'exprimer les surfaces courbes par des équations à trois variables dont j'ai appris qu'il étoit fait mention par occasion dans un mémoire du célèbre M. Bernoulli inséré dans les Actes de Leypsic [5].

In this quotation does "célèbre M. Bernoulli" mean James or John; I find nothing of this sort in James' collected works, nor in anything of John's except what I have already given, and that is not in the Acta Eruditorum.

4. LaHire and Parent. It is time to turn from those who might have invented analytic geometry in three dimensions but didn't, to those who did. First I take Philippe de LaHire. He stated definitely in 1679:

Je considère d'abord pour déterminer un point hors d'un plan, à l'égard d'une ligne droite déterminée sur ce plan, il faut trois conditions; la première est la grandeur $LA$ de la perpendiculaire menée du point $L$ au plan, la seconde la perpendiculaire $AB$ menée du point $A$ à la ligne donnée $OB$, et la troisième la partie $OB$ de cette ligne comprise entre un de ses points $O$ et le recontre $B$. C'est pourquoi je fais $OB = x, AB = y, LA = v$ qui sont trois inconnus.

He takes one specific problem, to find $L$ so that $a + OB = OL$. He sees geometrically

$$OL = \sqrt{x^2 + y^2 + v^2}; \quad a + x = \sqrt{x^2 + y^2 + v^2}$$

$$a^2 + 2ax = y^2 + v^2;$$

then he adds tamely enough

et comme il n'y a pas moyen de trouver d'autres équations pour faire évanouir quelques unes des inconnues, il s'ensuit que le problème est indéterminé [6].
So here we have a capable mathematician with all of three dimensional analytic geometry under his hand, but unable or unwilling to make anything of it save to calculate the distance of a point from the origin, and to note that one single condition will not locate completely a point in three-space.


It is hardly necessary to say that "affection" means equation; Parent is interested in the equations of surfaces. His first general problem is to determine the tangent plane. In his time mathematicians did not bother with the equations of tangents, but with their construction, and in the case of a plane curve, this came from the subtangent, which is the orthogonal projection on the x-axis of as much of the tangent as lies between the point of contact and the intersection with that axis; its value is \( ydx/dy \).

Parent begins with a sphere "pour exemple." He has a perfectly general method, but he begins with a sphere for he knows how to find its equation. Like LaHire he knows how to find the distance of a point from the origin. The radius of the sphere is \( r \), the coördinates of the center are, \( b, c, a; \) thus

\[
c^2 + y^2 - 2cy + b^2 + x^2 - 2bx + a^2 + z^2 - 2az = r^2.
\]

First holding \( y \) constant, and calmly disregarding algebraic signs, he has

\[
(a - z)dz = (b - x)dx; \quad \frac{zd\bar{x}}{dz} = \frac{a - z}{b - x}.
\]

Similarly,

\[
\frac{zd\bar{y}}{dz} = \frac{a - z}{c - y}.
\]

Clearly he has in mind the general formula, which we should write

\[
f(x, y, z) = 0, \quad \text{having subtangents} \quad -\frac{fs}{f_z}; \quad -\frac{fs}{f_y}.
\]

Parent next looks for those points on a surface where \( y \) takes a maximum or a minimum value. For a surface he takes a conchoid, discussed by L'Hospital, and allows one of the constants to vary; that is, he considers

\[
y^2 = \frac{z - x}{x} (b + x)^2.
\]

If we treat \( z \) as a constant, it follows that

\[
2ydy = (b + x)\left[ \frac{2(z - x)}{x} - \frac{(b + x)z}{x^2} \right] dx.
\]
For an extremal we have \( dy = 0 \); so
\[
2x^2 - zx + bz = 0.
\]

He does not recognize this as representing a cylinder, but sees that the projection on the \( xz \) plane is a hyperbola. By solving this for \( x \), and substituting in the equation of the surface, we get the projection on the \( yz \) plane.

Parent's third problem is to find the locus of the points of inflection in the planes, \( z = a \) constant. If we take \( x \) as the independent variable, we have at such points, in the notation of the time,
\[
dx = \text{a constant}, \quad ddy = 0.
\]

He needs a surface where \( y \) appears explicitly, so he takes
\[
y = \frac{z^3}{x^2 + az}.
\]

Keeping \( z \) constant,
\[
dy = \frac{-2xz^3dx}{(x^2 + az)^2},
\]
\[
ddy = (3x^2 - az) \frac{2z^2dx^2}{(x^2 + az)^3}.
\]

For a point of inflection we shall have
\[
3x^2 - az = 0.
\]

Parent does not go beyond this point; it is evident, however, that he had hold of a very real piece of mathematics for which he should deserve full credit.

5. Alexis Clairaut. I have already spoken of the work of the infant prodigy, Alexis Clairaut; it is time to speak further of his real contributions to three dimensional analytic geometry, even though he was late in time, 1731. Clairaut was inspired by the work of Descartes, which I have mentioned. He was interested in curves, not in surfaces, but he set up Cartesian coördinates exactly after the manner of LaHire and Parent, though he mentioned neither of these writers. His first theorem is that an equation involving three coördinates is a surface, because the intersection with any plane parallel to a coördinate plane is a curve. An equation of the first degree represents a plane, because the intersections with all planes, \( z = a \) constant, are straight lines parallel to one another, and the same is true for planes, \( y = a \) constant. He deals no further with planes but develops the formula for the square of the distance between two given points. He shows how to find the equation of a cone determined by a plane curve and a point outside of its plane, and how, when we have the equations of two surfaces, we may, by elimination, find the equation of the projection of their common curve on one of the coördinate planes.
Clairaut is happier when he passes to the application of the calculus to the study of curves. But here I should say that he does not go beyond first derivatives, or first differences, as he would have called them. The first and most fundamental problem is to find the tangent. The actual length of the tangent from the point of contact to the intersection with the plane \( z = 0 \) is \( z\sqrt{dx^2 + dy^2 + dz^2}/ds \). From this we can easily pass to various sorts of substrings. I note in passing that the square of \( dx \) is \( dx^2 \), but the square of \( x \) is \( xx \).

Most of the particular problems are not sufficiently interesting to be worth reproducing. The most amusing, perhaps, is that of finding the equation of the curve cut in the base plane \( z = 0 \) by a moving tangent to the space curve. We need the orthogonal projections on the \( x \) and \( y \) axes of so much of the tangent as runs from the point of contact to the base plane. These we get from the previous formulae for substrings, giving

\[
x' = x - \frac{zd}{dz}x; \quad y' = y - \frac{zd}{dz}y.
\]

From the finite and differential equations of the space curve we may find \( x, y, dx/dz, dy/dz \) in terms of \( z \). Eliminating, we have \( \phi(x', y') = 0 \).

The third section of Clairaut's work deals with the applications of the integral calculus to curve theory. He recognizes the distance formula

\[
ds = \sqrt{dx^2 + dy^2 + dz^2}.
\]

It is naturally hard to find a curve which can be rectified in this way. Here is one:

\[
y^2 - 2a^2 = \sqrt{9a^4x^2}; \quad az = y^2.
\]

Thus,

\[
x = \frac{(y^2 - 2a^2)^{1/2}}{3a^2}; \quad dx = \frac{\sqrt{y^2 - 2a^2ydy}}{a^2}; \quad dz = \frac{2ydy}{a};
\]

\[
ds = dy \frac{y^2 + a^2}{a^2};
\]

\[
s = \frac{y^2}{3a^2} + y + C.
\]

We can find the area of as much of one of the cylinders as is bounded by the curve and its projection on the \( yz \) plane from the integral \( \int x\sqrt{dy^2 + dz^2} \). The volume bounded by the two cylinders and the base plane is given by \( dv = yzdx \). He finds some other volumes of this sort.

The fourth section gives certain unconnected constructions of space curves. For instance, suppose that we have a compass of fixed opening \( r \) and one end is fixed at the point \( (a, b, c) \) of a given surface, what curve will the other point trace in the surface? Clearly

\[
f(x, y, z) = 0, \quad (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.
\]
Here is a more difficult problem: To find the equations of a curve along which a
given surface is touched by the tangents through a given point, \((a, b, c)\). We
have evidently

\[
\frac{dx}{x-a} = \frac{dy}{y-b} = \frac{dz}{z-c}.
\]

Here our modern notation gives the answer immediately. Besides the equa-
tion of the surface

\[f(x, y, z) = 0,\]

we have

\[f_x dx + f_y dy + f_z dz = 0,\]

\[f_x(x-a) + f_y(y-b) + f_z(z-c) = 0.\]

Clairaut's solution is, naturally, less compact. The lack of a suitable notation
for partial derivatives was a serious disadvantage to mathematicians of,
Clairaut's time, yet they were recognized by Newton. It is largely for this
reason that his book of 119 pages gives the impression of being very long for the
actual amount of mathematics involved, but when we remember that the
author was only sixteen when his memoir was first presented to the Académie
des Sciences, we can but wonder at his precocity.

6. Jakob Hermann. The year after the appearance of Clairaut's "Recherches"
another article, dealing with analytic geometry in three dimensions, but clearly
independent, first saw the light. The author was Jakob Hermann [7]. His
interest was in surfaces rather than in curves. He believed that the subject
had been neglected in the past because geometers were deterred by its apparent
proximity. He set up his axes exactly like those of Parent and Clairaut, though
neither of these authors is mentioned, and then gave a list of six surfaces he
meant to study. The first was

\[ax + by + cz - c^2 = 0.\]

This is shown to be a plane, by means of similar triangles. Next we have

\[z^2 - ax - by = 0.\]

This is a parabolic cylinder. The section in a plane, \(z=\) constant, is a straight
line, and all such lines are parallel. In the planes \(x=\) constant or \(y=\) constant,
we have a parabola. Let us find an extreme value for \(z\). Here we should have
\(dz/dx = dz/dy = 0\), but that involves \(a=b=0\), which is not the case, so there is
no extreme value which is fairly evident geometrically. To find a tangent plane
we seek the sub-tangents, as Parent did; that is,

\[
\frac{zd\alpha}{ds} = \frac{2(ax + by)}{a}; \quad \frac{zd\beta}{ds} = \frac{2(ax + by)}{b}.
\]
The next surface is
\[ z^2 - xy = 0. \]
This is a cone as it contains entirely a line connecting the origin with any other point of the surface. Sections perpendicular to the \( z \)-axis are rectangular hyperbolas; those in planes perpendicular to the other axes are parabolas. We find something more interesting in his fourth example,
\[ z^2 - ax^2 - bxy - cy^2 - ex - fy = 0. \]

Hermann remarks:

*Haec Aequatio est ad superficiem alcujs Conoidis, cujus basis exponitur ex aequatione*
\[ cx^2 + bxy + ax^2 + fy + ex = 0. \]
The wording here is interesting. "Conoidis" does not mean surface of revolution of Archimedes. Hermann deals separately with surfaces of revolution in a subsequent paragraph, and he refers explicitly to the Wallis wedge conoid. It seems certain that he was familiar with Wallis’ work, and took his definition of a conoid. He finds the extreme values of \( z \) by equating the two partial derivatives to 0; that is,
\[ 2ax + by + e = 0; \quad 2cy + bx + f = 0. \]
We solve these for \( x \) and \( y \), and substitute in the equation of the surface. He finds the tangent plane from the sub-tangents \( zdy/dz, zdx/dz \). This is so close to Parent’s work that one wonders whether he was not familiar with the French writer’s mathematics.

Hermann’s fifth surface gives something more difficult, namely,
\[ az^2 - bxz - cys + ey^2 = 0. \]
This again he recognizes as giving a cone, and he undertakes the none too easy problem of seeking for the planes of circular section. He takes \( u \) and \( t \) as rectangular Cartesian coordinates in such a plane, and expresses \( x, y, z \) in terms of these and of the constants determining the aspect of the plane; then he substitutes, and writes the conditions for a circle. He arrives finally at an equation of the third degree, as we should expect. The problem was not new. It had been handled in the 75th letter of Descartes, Third series, Ed. 1683, and in No. 441 of L’Hôpital’s *Traité analytique des sections coniques*.

Hermann’s last surface is
\[ u^2 - x^2 - y^2 = 0 \]
where \( u \) is a function of \( z \).

*Haec Aequatio ad omnis generis solidum spectat.*

He ends with a short study of geodesic curves, but acknowledges that the general problem is beyond him.

We may say that with Hermann the study of analytic geometry in three
dimensions was on a firm basis, and surfaces of the second order familiar objects. For that reason Cajori is clearly wrong when he writes of Euler: "He was the first to discuss the equation of the second degree in three variables, and to classify the surfaces represented by it." [8]. It is however true that he was the first to reduce such equations to canonical forms, in his *Analysin Infinitorum* of 1748. What a long time this is after Descartes’ *Géométrie* of 1637, where curves of the second degree were elaborately treated!

**References**

4. Johann Bernoulli, Opera omnia, Vol. IV, Lausanne, 1742.

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**MATHEMATICAL NOTES**

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**PARAMETRIC SOLUTIONS OF TWO MULTI-DEGREE EQUALITIES**

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1. **Introduction.** The notation

\[ A_1, A_2, \ldots, A_p^{n_1, n_2, \ldots, n_r} = B_1, B_2, \ldots, B_q \]

designates a so-called *multi-degreed equality* and means that the sum of the numbers on the left equals the sum of the numbers on the right for each of the \( r(n_1, n_2, \ldots, n_r) \) positive integral powers of the numbers.

We shall give here parametric solutions of the two multi-degreed equalities

\[(1) \quad A_1, A_2, A_3^{2,4} = B_1, B_2, B_3, \quad A_3 \neq A_1 + A_2, \quad B_3 = B_1 + B_2, \]

and

\[(2) \quad C_1, C_2, \ldots, C_7^{1,2,4,6,8} = D_1, D_2, \ldots, D_7. \]